বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2021

(Under CBCS Pattern)

Semester - VI

Subject: MATHEMATICS

Paper: DSE 4-T

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.
The figures in the margin indicate full marks.

MATHEMATICAL MODELLING

Answer any ***four*** questions from the following. 4×15=60

1. (a) State and Prove linearity and first shifted property for Laplace Transformation. 2+2

(b) If $\mathcal{L}(F(t)) = f(s)$, then prove that $\mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(u)\,du$, assuming that $\mathcal{L}\left[\frac{F(t)}{t}\right] \to 0$, as $s \to \infty$. 4

(c) State and Prove Convolution theorem of Laplace transformation. 3

(d) Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$ by convolution theorem of Laplace transformation. 4

2. (a) Obtain the power series solution of the equation

$$\left(1-x^2\right)y''(x)-2xy'(x)+p(p+1)y(x)=0$$, where p is a constant. 7

(b) Obtain the series solution of the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-4\right)y=0.$$ 8

3. (a) Solve the following differential equations using Laplace transformation

(i) $y''-6y'+15y=2\sin 3t$, $y(0)=-1$; $y'(0)=-4$. 4

(ii) $y''+5y'+6y=2e^{-t}$, $t\geq 0$, $y(0)=1$, $y'(0)=0$ 4

(b) Solve $2x^2(x-1)y''(x)+x(3x+1)y'(x)-2y=0$ in series, near $x=\infty$. 7

4. Consider a small harbor with unloading facilities for ships. Only one ship can be unloaded at any one time. Ships arrive for unloading of cargo at the harbor, and the time between the arrival of successive ships varies from 15 to 145 min. The unloading time required for a ship depends on the type and amount of cargo and varies from 45 to 90 min. Now, answers the following questions:

(a) What are the average and maximum times per ship in the harbor ?

(b) If the waiting time for a ship is the time between its arrival and the start of unloading, what are the average and maximum waiting times per ship ?

(c) What percentage of the time ar ethe unloading facilities idle ?

(d) What is the length of the longest queue ?

Where data for five ships are provided in the following table:

	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5
Time between successive ships	20	30	15	120	25
Unloading time	55	45	60	75	80

15

5. (a) Using Monte Carlo simulation, write an algorithm to calculate that part of the volume of an ellipsoid $\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{8}\le 16$ that lies in the first octant $x>0,\ y>0,\ z>0$. 7

(b) Describe with examples linear congruence method to generate random numbers. 4

(c) Use the middle-square method to generate

(i) 5 random numbers using $x_0 = 1009$

(ii) 5 random numbers using $x_0 = 653217$. 2+2

6. Describe the steps of simplex method for optimization and solve the following problem by using simplex method

Maximum $z = 6x + 4y$

Subject to
$$-x + y \le 12$$
$$x + y \le 24$$
$$2x + 5y \le 80$$
$$x \ge 0,\ y \ge 0$$

7+8

7. Why is sensitivity analysis important in linear programming ? 5+5+5

8. Solve the following linear programing problem graphically and algebraically

Maximum $z = 25x + 30y$

Subject to
$$20x + 30y \le 690$$
$$5x + 4y \le 120$$
$$x,\ y \ge 0$$

7+8

Or

DIFFERENTIAL GEOMETRY

Answer any ***four*** questions from the following. $4\times15=60$

1. (a) Let γ be a regular curve in $\mathbb{R}^3$ with no where vanishing curvature (so that the torsion τ of γ is defined). Prove that the image of γ is contained in a plane if and only if τ is zero at every point of the curve.

   (b) Let γ be a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Apply Serret-Frenet equations to prove that γ is a part of circle.

   (c) Compute the torsion of the circular helix $\gamma(\theta)=(a\cos\theta,\ a\sin\theta,\ b\theta)$ 5+5+5

2. (a) Find the equation of the osculating plane at a general point on the curve given by

   $\vec{r}=(u,\ u^2,\ u^3)$

   (b) Prove that the principal normals of a curve are binormals of another, if the relation $a(\kappa^2+\tau^2)=b\kappa$ holds where a, b are constants.

   (c) Prove that the locus of the centre of curvature is an evolute only when the curve is plane. 5+5+5

3. (a) Let $\vec{r}=\vec{r}(u,\ v)$ be the equation of a surface. Prove that the 1st Fundamental form $E\,du^2+2F du\,dv+G\,dv^2$ remains invariant if the parameters u, v are transformed to the parameters u, v are transformed to the paramters u', v' by the formula $u'=\varphi(u,v)$, $v'=\psi(u,v)$.

   (b) Calculate the fundamental magnitudes for the surface of revolution

   $x=u\cos v,\ \ y=u\sin v,\ \ z=f(u)$. 8+7

4. (a) Obtain the differential equation of the lines of curvature through a point on the surface $F(x,y,z)=0$.

   (b) State and Prove Rodrigue's Formula.

(c) Find the value of (i) First Curvature (ii) Gaussian Curvature at any point of the right helicoid $\vec{r} = (u\cos v,\ u\sin v, av)$. 5+7+3

5. (a) Show that the value of Gaussian Curvature is independent of the particular parametric respresentation chosen.

(b) Prove that binormal of asymptotic line is the normal to the surface.

(c) Find the asymptotic lines on the surface $z = y\sin x$. 8+3+4

6. (a) Prove that a necessary and sufficient condition that a curve on a developable surface be a geodesic is that the surface be the rectifying developable of the curve.

(b) A particle is constrained to move on a smooth surface under no force except the normal reaction. Show that its path is a geodesic.

(c) Show that the curves $u + v = \text{constant}$ are geodesics on a surface with metric.

$(1+u^2)du^2 - 2uv\, du\, dv + (1+v^2)dv^2$. 7+4+4

7. (a) State and prove Clairaut's Theorem.

(b) Prove that geodesics on right circular cylinders are helices.

(c) Using Gauss Bonnet theorem prove that on a surface of negative curvature two geodesics cannot meet in two points and enclose a simply connected area. 5+5+5

8. (a) Prove that the necessary and sufficient condition for a curve on a sphere to be geodesic is that the curve is a great circle.

(b) If κ & τ are the curvature and torsion of a geodesic prove that

$\tau^2 = (\kappa - \kappa_a)(\kappa_b - \kappa)$ where κ_a & κ_b are principal curvatures.

(c) If two surfaces touch each other along a curve and if the curve is a geodesic on one surface then prove that it is a geodesic on the other also. 7+4+4

Or

BIO MATHEMATICS

Answer any ***four*** questions from the following. $4\times15=60$

1. (a) What are the basic assumptions of the two species prey-predator population model ? With these assumptions one imporvement in the prey-predator model is to modify the equation for the prey so that it has the form of a logistic equation in the absence of the predator. 4

   (b) What will be the form of the system of equations modelling this situation ? 4

   (c) Determine all critical points of the system and discuss their nature and stability. 7

2. A simple Lotka-Volterra two species competition model is given by the following differential equations :

$$\frac{dx}{dt} = x\left(a_1 - b_{11}x - b_{12}y\right)$$

$$\frac{dy}{dt} = y\left(a_2 - b_{21}x - b_{22}y\right)$$

with initial condition $x\left(t_0\right) = x_0$ and $y\left(t_0\right) = y_0$,

where $a_1\left(>0\right)$, $b_{11}\left(>0\right)$, $a_2\left(>0\right)$, $b_{22}\left(>0\right)$ are the logistic parameters for the first and second species and $b_{12}\left(>0\right)$, $b_{21}\left(>0\right)$ arise from the inhibiting effect on the growth of the first species due to the presence of the second species and from the effect on the growth of the second species due to the presence of the first species.

   (a) Discuss the stability of the equilibrium position by finding the secular equation of the above model. 7

   (b) Draw the direction diagrams in the phase plane by define the term phase plane. 5

   (c) What is Lypunov function ? Find the Lypunov function, if it exists for above model. 3

3. (a) What are the basic assumptions of SIR (Susceptible-Immigration-Removal) epidemic model ? 3

(b) Formulate differential equation and find steady state solution of SIR model. 6

(c) Discuss the stability of the steady state solution of that. 6

4. A reaction diffusion model is given by

$$\frac{\partial M_1}{\partial t} = r\left(a - M_1 + M_1^2 M_2\right) + \frac{\partial^2 M_1}{\partial x^2}$$

$$\frac{\partial M_2}{\partial t} = r\left(b - M_1^2 M_2\right) + D\frac{\partial^2 M_2}{\partial x^2}$$

where M_1 and M_2 are morphogen concentrations. All the parameters r, a, b, D are positive constants.

(a) Explain the model and find the non-zero homogeneous steady state. 5

(b) Show that the condition of stability (without diffusion) is $b - a - (a+b)^3 < 0$. 5

(c) Linearize the system about the non-zero steady state and find the condition for diffusive instability. 5

5. (a) Deduce Fisher's equation for spreading of genes. 5

(b) What are the additional restrictions on Fisher's problem for traveling wave solution ? 5

(c) Analyze the phase plane and local stability on Fisher's problem. 5

6. (a) In a hypothetical population

$\frac{\text{Birth rate}}{x} = \left(\frac{1}{2} - \frac{1}{800}x\right)$ and $\frac{\text{Death rate}}{x} = \left(\frac{1}{4} + \frac{1}{200}x\right)$. We assume the population to be closed to the outside. Find the size x of the population as a function of the time t and also find the population at the point of inflexion. Given that $x_0 = 2$. 8

(b) Explain under what condition the logistic model is extended to Allee effect and Gompertz law. 7

7. (a) Define age structured population model. 7

(b) Suppose a patient is given a drug (Drug delivery problem) to treat some infection. The amount of drug in the patient's bloodstream decreases at the rate of 50% per hour. To sustain the drug to a certain level, an injection is given at the end of each hour that increases the amount of drug in the bloodstream by 0.2 units.

(i) Formulate a dynamical model which describe the above scenario. 4

(ii) Find the equilibrium solution and sketch the graph of drug amount in the blood against time. 4

8. (a) What are several assumptions that lead to form Nicholoson-Bailey (host-parasitoid) model ? 4

(b) What are the equilibrium solutions of Nicholoson-Bailey (host-parasitoid) model ? 4

(c) Discuss the stability of each equilibrium solution. 7